

# Measurable Categories

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**Abstract:** We develop the theory of categories of measurable fields of Hilbert spaces and bounded fields of operators. We examine classes of functors and natural transformations with good measure theoretic properties, providing in the end a rigorous construction for the bicategory used in [4] and [3] as the basis for a representation theory of (Lie) 2-groups. Two important technical results are established along the way: first it is shown that all invertible additive bounded functors (and thus *a fortiori* all invertible \*-functors) between categories of measurable fields of Hilbert spaces are induced by invertible measurable transformations between the underlying Borel spaces and second we establish the distributivity of Hilbert space tensor product over direct integrals over Lusin spaces with respect to  $\sigma$ -finite measures. The paper concludes with a general definition of measurable bicategories.

## 1 Introduction

One of the vexing problems in the algebraic approach to quantum topology and related work on algebraic models for quantum gravity has been a lack of suitable examples of the algebraic structures. Just as classical examples of tensor categories were “too commutative” to have any but the most trivial relation to classical knots and three manifolds, so known examples of algebraic structures of the type expected at a formal level to be related to the structure of 4-manifolds appear to be “too commutative” to detect anything other than homeomorphism type.

The desire to find sufficiently non-commutative examples of such structures (for example, monoidal bicategories with appropriate dualities) has been one motivation for the work in higher dimensional algebra done in the past decade. Another has been the suspicion that symmetry groups may be inadequate expressions of the symmetries needed to formulate a quantum theory of gravity.

The present work is intended to address difficulties in one line of development in this direction: the representation theory of categorical groups (or, as they are called when considered as a type of bicategory with one object, 2-groups<sup>1</sup>). This representation theory is not developed in the present paper beyond stating the definition to give an example of a general notion of measurable bicategory. Rather this is the subject of [4] and [3].

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<sup>1</sup>Not to be confused with the use of “2-groups” in reference to 2-torsion in a finite group

The difficulties in representing 2-groups with infinite sets of objects in any of the versions of 2-VECT considered by Kapranov and Voevodsky [7] (cf. also [1], [2]) also are analogous to the difficulties in representing non-compact groups in the category of finite-dimensional vector spaces. The natural way to overcome them is the same: move to a setting built out of measurable spaces rather than finite sets.

Just as in 2-VECT one has families of vector-spaces indexed over finite sets, so we need a setting where we have families of Hilbert spaces indexed over measure spaces.

The appropriate ideas have already been considered in the context of functional analysis and the (unitary) representation theory of non-compact groups. There the generalization of direct sum decomposition theorems required the introduction of a measure-theoretic analogue: the direct integral.

Although the constructions in Sections 2 through 4 are perfectly general and will work for any measurable space satisfying the mild technical hypothesis that points are measurable, beginning with section 5, where we will need to invoke results of Maharam [8] (cf. also [6]) on disintegrations of measures, we will require the hypothesis that the measurable spaces in question are the Borel space associated to a Lusin space (the image of a separable metric space under a continuous bijection) whose points are Borel sets.<sup>2</sup> Beginning in Section 4, although the definitions are applicable to general measures, our results will require the use of the Radon-Nikodym Theorem and the Lebesgue Decomposition Theorem. Thus we will assume throughout that all measures are totally  $\sigma$ -finite. Finally, we will need at one point to assume that the  $L^2$  spaces for all measure spaces considered are separable Hilbert spaces. Observe that this rather long list of technical hypotheses is satisfied by among others the Haar measure on any finite dimensional Lie group and by any measure on a Lusin space concentrated at a countable set of points.

## 2 Categories of Measurable Fields of Hilbert Spaces

The rich structures associated to the direct integral construction appear never to have been examined from a categorical point of view, although all the necessary ingredients are there. Indeed the first two definitions are variations of notions found in Takesaki [9]:

**Definition 1** *A measurable field of Hilbert spaces  $\mathcal{H}$  on a Borel space  $(X, S)$  is a pair  $(\mathcal{H}_x, \mathcal{M}_{\mathcal{H}})$ , where  $\mathcal{H}_x$  is an  $X$ -indexed family of Hilbert spaces, and  $\mathcal{M}_{\mathcal{H}} = \mathcal{M}$  is a linear subspace of  $\prod_{x \in X} \mathcal{H}_x$  (the product as vector-spaces) satisfying*

1.  $\forall \xi \in \mathcal{M} \ x \mapsto \|\xi(x)\|_x$  is measurable,

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<sup>2</sup>This is equivalent to an odd separation axiom stronger than  $T_0$ , but weaker than  $T_1$ , “Freyd’s Favorite Separation Axiom”: every point is locally closed, that is, every point admits an open neighborhood in which it is a closed set.

2. For all  $\eta \in \prod_{x \in X} \mathcal{H}_x$  the measurability of  $x \mapsto \langle \eta(x) | \xi(x) \rangle_x$  for all  $\xi \in \mathcal{M}$  implies  $\eta \in \mathcal{M}$ ,
3.  $\exists \{\xi_i\}_{i=1}^\infty \subset \mathcal{M}$  such that  $\{\xi_i(x)\}_{i=1}^\infty$  is dense in  $\mathcal{H}_x$  for all  $x \in X$ .

An almost measurable field of Hilbert spaces  $\mathcal{H}$  on a Borel space  $(X, S)$  is a pair  $(\mathcal{H}_x, \mathcal{M}_{\mathcal{H}})$  as above, satisfying conditions 1 and 2, but not necessarily condition 3.

One thing which should be noted immediately is that condition 1, together with a polarization argument, shows that condition 2 is really bidirectional:  $\xi \in \mathcal{M}$  if and only if for all  $\zeta \in \mathcal{M}$ ,  $x \mapsto \langle \xi(x) | \zeta(x) \rangle_x$  is measurable.

In what follows, we will assume w.l.o.g. that the sequence  $\{\xi_i\}$  verifying condition 3. includes the 0 section. This assumption simplifies some constructions.

It might seem logical to use “measurable field” to refer to what we have called an almost measurable field. There is, however, reason beyond adherence to established terminology not to do so. Example 4 shows that although the measurability conditions are maintained in the definition of an almost measurable field, the lack of the countability or separability condition allows non-measurable pathologies.

**Definition 2** A measurable field of bounded operators  $\phi$  from  $\mathcal{H}$  to  $\mathcal{K}$ , for  $\mathcal{H}$  and  $\mathcal{K}$  (almost) measurable fields of Hilbert spaces is an  $X$ -indexed family of bounded operators  $\phi_x \in B(\mathcal{H}_x, \mathcal{K}_x)$  such that  $\xi \in \mathcal{M}_{\mathcal{H}}$  implies  $\phi(\xi) \in \mathcal{M}_{\mathcal{K}}$ , where  $\phi(\xi)_x = \phi_x(\xi_x)$ .

A measurable field of bounded operators is bounded if the real valued function  $x \mapsto \|\phi_x\|_x$  is bounded.

A measurable field of bounded operators is essentially bounded with respect to a measure  $\mu$  on  $(X, S)$  if  $x \mapsto \|\phi_x\|_x$  is in  $L^\infty(X, \mu)$ . (Here  $\|\cdot\|_x$  denotes the operator norm on  $B(\mathcal{H}_x, \mathcal{K}_x)$ .)

Classically measure spaces are considered, and it is thus more natural to work with essentially bounded fields, as two field of operators which differ on a set of measure zero will induce the same operator between direct integrals. We, however, are working in a setting where different measures will be considered on the same Borel space, and are thus obliged to work with bounded fields, “measure zero” having no fixed meaning when one changes measures.

We can then organize these into a category:

**Definition 3** The category of measurable fields of Hilbert spaces on  $(X, S)$  has as objects all measurable fields of Hilbert spaces on  $(X, S)$  and as arrows all bounded fields of operators on  $X$ . Source, target, identity arrow and composition are obvious. We denote this category by  $\mathbf{Meas}(X, S)$ .

Similarly, the category of almost measurable fields of Hilbert spaces on  $(X, S)$  has as objects all almost measurable fields of Hilbert spaces on  $(X, S)$  and as arrows all bounded fields of bounded operator between them. We denote this category by  $\mathbf{AlMeas}(X, S)$ .

It will be important in what follows to organize these families of categories into 2-categories by introducing suitable functors and natural transformations between them.

Before we do this, however, we consider an example which explains why the name “measurable field” is properly applied to the classical notion rather than to the more general notion, and derive some elementary properties of the categories themselves.

**Example 4** Consider the almost measurable field of Hilbert spaces on  $(\mathbb{R}, S)$ , the real line with the Borel structure of all Borel measurable sets, defined as follows: Let  $X$  be a non-measurable subset of  $\mathbb{R}$ , and let  $\{A_\lambda\}$  be a (necessarily uncountable) set of measurable sets such that  $\cup_\lambda A_\lambda = X$ . Now, consider the field of Hilbert spaces

$$\mathcal{H}_x = \begin{cases} \mathbb{C} & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

with  $\mathcal{M}$  given by  $\{\zeta|x \mapsto \langle \zeta | \xi_\lambda \rangle \text{ is measurable for all } \lambda\}$ , where  $\xi_\lambda$  is the section which is 1 on each fiber over  $A_\lambda$  and 0 otherwise.

Note the pathology present in the example just given: although the sections in  $\mathcal{M}$  all have measurable fiberwise norms, and give measurable functions when their fiberwise scalar products are taken, the support of the field is non-measurable. This is so even though the fibers are all separable.

**Proposition 5** For any Borel space  $(X, S)$ , the categories  $\mathbf{Meas}(X, S)$  and  $\mathbf{AlMeas}(X, S)$  are  $\mathbb{C}$ -linear additive  $C^*$ -categories.

**proof:** It is easy to see that the hom-set, equipped with fiberwise addition, and multiplication by scalars are modules over the algebra of bounded functions on  $X$ , and thus, *a fortiori* vectorspaces over  $\mathbb{C}$ . Moreover they are equipped with a norm  $\|\phi\| = \sup_{x \in X} \|\phi_x\|_x$  where  $\|\cdot\|_x$  is the operator norm of the bounded operator at  $x$ .

The field with constant fiber 0 is plainly a zero object. It thus remains to show that the category admits biproducts.

As observed in the Appendix,  $\mathbf{Hilb}$  is an additive category. It follows that  $\mathbf{Hilb}^X$  is as well. Now observe that there are forgetful functors from  $\mathbf{Meas}(X, S)$  and  $\mathbf{AlMeas}(X, S)$  to  $\mathbf{Hilb}^X$ . We claim that these functors creates biproducts, in the sense that the biproduct  $\{\mathcal{H}_x \oplus \mathcal{K}_x\}$  of the underlying  $X$ -indexed families of Hilbert spaces  $\{\mathcal{H}_x\}$  and  $\{\mathcal{K}_x\}$  has the structure of a(n almost) measurable field of Hilbert spaces, and that the projections and inclusions are measurable fields of operators.

The fibers of the biproduct are the direct sum  $\mathcal{H}_x \oplus \mathcal{K}_x$  of the underlying vectorspaces, with the sum of the scalar products on the summands as scalar product (see the Appendix). We can then form  $\mathcal{M}_{\mathcal{H} \oplus \mathcal{K}}$  by taking the closure of the set  $G = \{(\eta, 0) | \eta \in \mathcal{M}_{\mathcal{H}}\} \cup \{(0, \kappa) | \kappa \in \mathcal{M}_{\mathcal{K}}\}$  under condition 2. of the definition of (almost) measurable fields (cf. [9]).

Now, observe that if  $\{\eta_i\}$  and  $\{\kappa_i\}$  are fundamental sequences for  $\mathcal{H}$  and  $\mathcal{K}$  respectively, then the sequence  $\{(\eta_i, \kappa_j)\}$  satisfies the required condition that  $\{(\eta_i(x), \kappa_j(x))\}$  is dense in  $\mathcal{H}_x \oplus \mathcal{K}_x$ , and moreover is in  $\mathcal{M}_{\mathcal{H} \oplus \mathcal{K}}$ . since the scalar products of elements with elements of  $G$  are plainly measurable

Now it is easy to see that the inclusions preserve measurable sections: for example, for the first inclusion  $\langle(\zeta, 0)|(\eta, 0)\rangle = \langle\zeta|\eta\rangle$ , while  $\langle(\zeta, 0)|(0, \kappa)\rangle = 0$  and thus  $(\zeta, 0)$  is measurable whenever  $\zeta$  is.

For the projections, consider a section  $(\zeta, \lambda)$ . By construction it is measurable whenever the fiberwise scalar products with each of the  $(\eta, 0)$ 's and each of the  $(0, \kappa)$ 's are measurable functions. But taking scalar products with these reduces to taking scalar products of one of the summands alone, thus implying that each of the summands  $\zeta$  and  $\lambda$  are measurable.

For boundedness, it follows from the construction of the biproduct on the direct sum in **Hilb** that the inclusions (resp. projections) are norm preserving (resp. norm decreasing) in each fiber, and thus, taking suprema are norm preserving (resp. norm decreasing).

It thus remains to show that the hom-sets are complete with respect to the norm, that  $\|\phi(\psi)\| \leq \|\phi\|\|\psi\|$  and  $\|\phi(\phi^*)\| = \|\phi\|$ . All three follow from the corresponding result for **Hilb** once it is observed that equations and non-strict inequalities are preserved by suprema, and that Cauchy sequences of bounded fields of operators necessarily give Cauchy sequences in each fiber since the norm is the supremum of the operator norms in each fiber. •

To further develop the theory, we need some notions native to categories of the form **Meas** $(X, S)$  and **AlMeas** $(X, S)$ .

**Definition 6** *The support of (an almost) measurable field of Hilbert spaces  $\mathcal{H}$  on a Borel space  $(X, S)$  is the set  $\text{supp}(\mathcal{H}) = \{x \in X | \mathcal{H}_x \not\cong 0\}$ .*

*The support of a measurable section  $\xi \in \mathcal{M}_{\mathcal{H}}$  is the set  $\text{supp}(\xi) = \{x \in X | \xi(x) \neq 0\}$ . Note by taking norms that it is necessarily a measurable set.*

*The support of a field of bounded operators  $B$  is the set  $\text{supp}(B) = \{x \in X | B(x) \neq 0\}$ .*

**Definition 7** *If  $\mathcal{H}$  (resp.  $\xi, B$ ) is a(n almost) measurable field of Hilbert spaces (resp. a measurable section, a measurable field of operators) on  $(X, S)$ , and  $A \in S$ , then the restriction of  $\mathcal{H}$  (resp.  $\xi, B$ ) to  $A$ , denoted  $\mathcal{H}|_A$  (resp.  $\xi|_A, B|_A$ ), is given by*

$$\mathcal{H}|_{Ax} = \begin{cases} \mathcal{H}_x & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

*with  $\mathcal{M}_{\mathcal{H}|_A} = \mathcal{M}_{\mathcal{H}} \cap \prod_x \mathcal{H}|_{Ax}$ , where we identify  $\mathcal{H}|_{Ax}$  with a subspace of  $\mathcal{H}_x$ , either the entire Hilbert space or 0 (resp.*

$$\xi|_A(x) = \begin{cases} \xi(x) & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases},$$

$$B|_A(x) = \begin{cases} B(x) & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}.$$

In the measurable case, we need to see that if  $\{\xi_i\}$  is a fundamental sequence for  $\mathcal{H}$ , then  $\{\xi_i|_A\}$  is a fundamental sequence for  $\mathcal{H}|_A$ . This will follow directly from

**Lemma 8** *For any measurable set  $A$  and any (almost) measurable field of Hilbert spaces  $\mathcal{H}$ ,  $\xi \in \mathcal{M}_{\mathcal{H}}$  implies  $\xi|_A \in \mathcal{M}_{\mathcal{H}}$ .*

**proof:** We show that  $f(x) = \langle \xi|_A | \zeta \rangle$  is measurable for any  $\zeta \in \mathcal{M}_{\mathcal{H}}$ . Let  $g(x) = \langle \xi | \zeta \rangle$ . Recall that a real-valued function  $\phi$  is measurable if for any measurable set  $M \subset \mathbb{R}$  the set  $N(\phi) \cap \phi^{-1}(M)$  is measurable, where  $N(\phi) = \{x | \phi(x) \neq 0\}$ . Observing that  $f$  and  $g$  agree on  $A$ , but that  $f$  is constant 0 on  $\neg A$ , we have that  $N(f) = N(g) \cap A$ , while  $f^{-1}(M) = g^{-1}(M) \cap A$  if  $0 \notin M$  and  $f^{-1}(M) = g^{-1}(M) \cup \neg A$  if  $0 \in M$ .

In either event the intersection  $N(f) \cap f^{-1}(M)$  is  $N(g) \cap g^{-1}(M) \cap A$ , which is measurable since both  $A$  and  $N(g) \cap g^{-1}(M)$  are. •

Restrictions in general provide objects  $\mathbf{Meas}(X, S)$  with direct sum decompositions:

**Theorem 9** *If  $\mathcal{H}$  is a measurable field of Hilbert spaces on  $(X, S)$  and  $A \in S$ , then*

$$\mathcal{H} \cong \mathcal{H}|_A \oplus \mathcal{H}|_{\neg A}$$

**proof:** This follows almost immediately from three observations about  $|_A$  on sections: first  $|_A$  is idempotent, second  $|_A|_{\neg A} = 0$ , and third for any section  $\xi$ , we have  $\xi = \xi|_A + \xi|_{\neg A}$ . •

**Definition 10** *A measurable field of Hilbert spaces  $\mathcal{H}$  on  $(X, S)$  is a partial measurable line bundle if for all  $x \in X$  either  $\mathcal{H}_x \cong \mathbb{C}$  or  $\mathcal{H}_x \cong 0$ .*

We have already seen a pathology which can occur when the corresponding notion is considered in the almost measurable setting. We now establish that it does not occur for partial measurable line bundles. Indeed we have

**Theorem 11**  *$\text{supp}(-)$  induces a canonical bijection between the isomorphism classes of partial measurable line bundles on  $(X, S)$  and the collection  $S$  of measurable sets.*

**proof:** We proceed in two stages: first we show that  $\text{supp}(-)$  takes values in the measurable sets, and then we construct an inverse.

Consider a partial measurable line bundle  $\mathcal{H}_x, \mathcal{M}, \{\xi_i\}$ . Now it is clear that the union of the supports of any family of measurable sections of  $\mathcal{H}$  is contained in the support of  $\mathcal{H}$ , as fibers outside the support are 0. It thus suffices to see that there is a set of sections whose support contains that of  $\mathcal{H}$  and is measurable.

Consider the fundamental sequence  $\{\xi_i\}$ . By the density condition, given any non-zero element  $v$  of the fiber  $\mathcal{H}_x$ , there is a  $\xi_i$  such that  $\|v - \xi_i(x)\| <$

$\frac{1}{2}\|v\|$ , and thus such that  $\xi_i(x) \neq 0$ . From this it follows immediately that  $\bigcup_i \text{supp}(\xi_i) = \text{supp}(\mathcal{H})$ .

But, the  $\text{supp}(\xi_i)$ 's are a countable set of measurable sets in a Borel space, and thus their union is measurable.

To construct an inverse to  $\text{supp}(-)$ , we proceed as follows: Given a measurable set  $A \subset X$ , we may form a partial measurable line bundle with support  $A$  by taking the restriction of the constant measurable field  $\mathbb{C}$ .

It thus suffices to show that any partial measurable line  $\mathcal{H}$  bundle with support  $A$  is isomorphic to  $\mathbb{C}|_A$ .

To construct the isomorphism, we first construct a measurable section of  $\mathcal{H}$  whose support coincides with that of  $\mathcal{H}$ :

$$\xi_{total} = \sum_{i=1}^{\infty} \xi_i|_{\neg \bigcup_{j<i} \text{supp}(\xi_j)}$$

As usual, to see that this sum lies in  $\mathcal{M}$ , we consider the fiberwise scalar product with a test section  $\zeta \in \mathcal{M}$ . As the supports are disjoint, it is easy to see that

$$\langle \xi_{total} | \zeta \rangle = \sum_{i=1}^{\infty} \langle \xi_i |_{\neg \bigcup_{j<i} \text{supp}(\xi_j)} | \zeta \rangle$$

By Lemma 8 each of the summands is measurable, however, it is quite easy to construct examples in which the convergence of the sequence of partial sums is not uniform a.e.

To see that the limit is, in fact, measurable, observe that the disjointness of the supports of the summands implies that for any Borel set  $M$ , the sets  $N(f_i) \cap f_i^{-1}(M)$  are disjoint, where  $f_i$  is the  $i^{th}$  summand in the series of scalar products, and moreover, letting  $f(x) = \langle \xi_{total}(x) | \zeta(x) \rangle_x$ , that  $N(f) \cap f^{-1}(M)$  is their (disjoint) union. Note that the  $N(f_i) \cap f_i^{-1}(M)$  are measurable by Lemma 8, and thus, their union is.

Having established that  $\xi_{total}$  is a measurable section of  $\mathcal{H}$  with support equal to that of  $\mathcal{H}$ , observe that if  $\mathcal{H}_x$  is non-zero, then the singleton  $\{\xi_{total}(x)\}$  is a basis. This suggests that we should have an isomorphism from  $\mathcal{H}$  to  $\mathbb{C}|_A$  given by the field of operators  $\psi$  which maps  $\xi_{total}(x)$  to  $1|_A$ . The difficulty is, that as it stands neither this nor its inverse need be bounded fields of operators.

To correct this, we replace  $\xi_{total}$  with  $\xi_{normal} = \frac{1}{\|\xi_{total}(x)\|_x} \xi_{total}(x)$ . This is a measurable section by Lemma 13 below. We can now give an isomorphism from  $\mathcal{H}$  to  $\mathbb{C}|_A$  by the field of operators  $\phi$  which maps  $\xi_{normal}(x)$  to  $1|_A$ .

The fields of operators  $\phi$  and  $\phi^{-1}$  in fact preserve the scalar product in each fiber, and are thus plainly measurable and bounded. •

Applied to any measurable field of Hilbert spaces  $\mathcal{H}$  the construction just given shows that there is a measurable section  $\xi_{normal}$  with the same support as  $\mathcal{H}$  and norm 1 in every non-zero fiber..

This observation, together with Proposition 15 will show

**Theorem 12** *For every measurable field of Hilbert spaces  $\mathcal{H}$ , there exists a partial measurable line bundle  $\mathcal{L}$  which is a direct summand of  $\mathcal{H}$  and has the same support as  $\mathcal{H}$ .*

First observe

**Lemma 13** *If  $\xi$  is a measurable vector field for some measurable field  $\mathcal{H}$  on  $(X, S)$  and  $\phi : X \rightarrow \mathbb{R}$  is a measurable function, then  $\phi\xi$ , the section given at  $x$  by  $\phi(x)\xi(x) \in \mathcal{H}_x$  is measurable.*

**proof:** Let  $\zeta \in \mathcal{M}_{\mathcal{H}}$ , then

$$\langle \phi(x)\xi(x) | \zeta \rangle_x = \phi(x) \langle \xi(x) | \zeta \rangle_x$$

by linearity in each fiber. But as products of measurable functions are measurable, the latter defines a measurable function on  $X$ , and by condition 2,  $\phi\xi$  is a measurable vector field. •

From this, together with the observation that sums, products and reciprocals of measurable real-valued functions are measurable, we see

**Proposition 14** *If the Gram-Schmidt process is applied (fiberwise) to a set of measurable vector fields, the result is a set of measurable vector fields.*

**Proposition 15** *If  $\{\xi^{(i)}\}$  for  $i = 1, \dots, n$  is a finite set of measurable vector fields in  $\mathcal{H}$  and  $\mathcal{G}_x = \text{span}\{\xi^{(i)}(x)\}$  then  $\mathcal{G} = (\mathcal{G}_x, \mathcal{M}_{\mathcal{H}} \cap \prod_x \mathcal{G}_x)$  is a measurable field of Hilbert spaces, and moreover a direct summand of  $\mathcal{H}$  with complementary summand given by  $\mathcal{G}^\perp = (\mathcal{G}_x^\perp, \mathcal{M}_{\mathcal{H}} \cap \prod_x \mathcal{G}_x^\perp)$ .*

**proof:** By Proposition 14 it follows that the fiberwise orthogonal projections onto  $\mathcal{G}$  and  $\mathcal{G}^\perp$  preserve measurable sections. Now, observe in both  $\mathcal{G}$  and  $\mathcal{G}^\perp$  the image of the fundamental sequence of  $\mathcal{H}$  under the orthogonal projections may be taken as the fundamental sequence for the subfield. Observe also that in either case, since the projection is idempotent, it follows that the image of the measurable sections is precisely  $\mathcal{M}_{\mathcal{H}} \cap \prod_x \mathcal{G}_x$  or  $\mathcal{M}_{\mathcal{H}} \cap \prod_x \mathcal{G}_x^\perp$  respectively.

We have thus established that conditions 1 and 3 hold in both  $\mathcal{G}$  and  $\mathcal{G}^\perp$ , and that once condition 2 is shown, that the orthogonal projections and inclusions are measurable fields of operators. The required equational condition for the direct sum decomposition follows from fiberwise condition in **Hilb**. The inclusions are plainly norm preserving, and it follows from this, the equational condition, and orthogonality that the projections are norm decreasing.

To establish condition 2, observe that any measurable vector field  $\zeta$  decomposes as a sum of its projections onto measurable vector fields  $\zeta_\parallel \in \mathcal{G}$  and  $\zeta_\perp \in \mathcal{G}^\perp$ . By orthogonality, it follows that for a section  $\xi$  of  $\mathcal{G}$  (resp.  $\mathcal{G}^\perp$ ) the scalar product  $\langle \xi | \zeta \rangle$  is equal to  $\langle \xi | \zeta_\parallel \rangle$  (resp.  $\langle \xi | \zeta_\perp \rangle$ ). From this and condition 2 for  $\mathcal{H}$ , condition 2 for the subfields follows immediately. •



### 3 Bounded Invertible Additive Functors

Since the primary motivation for this work is the construction of a suitable setting for the representation theory of 2-groups, we begin considering a family of functors sufficient for the construction of the representations themselves. In Section 5 we will construct the larger family of functors which will be needed for the intertwiners in the theory developed in [4].

**Definition 16** *An invertible additive functor is a functor  $F$  between two additive categories, which admits an inverse functor (up to natural isomorphism)  $G$  such that both  $F$  and  $G$  preserve the addition of parallel maps, the zero object (up to canonical isomorphism) and the biproducts (up to canonical isomorphism).*

**Definition 17** *A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between  $C^*$ -categories is bounded if there exists a constant  $N > 0$  such that for all  $f : X \rightarrow Y \in \text{Arr}(\mathcal{C})$*

$$\|F(f)\| \leq N\|f\|.$$

Observe that  $*$ -functors are bounded, being, in fact, norm decreasing (cf. [5]).

**Definition 18** *A natural transformation  $t$  between bounded functors is bounded if  $\|t\| = \sup_{X \in \text{Ob}(\mathcal{C})} \|t_X\| < +\infty$ .*

Our goal in this section is to characterize bounded invertible additive functors between categories of measurable fields of Hilbert spaces in terms of the Borel space structures on the source and target. In particular, we will show that any bounded invertible additive functor between categories of the form  $\mathbf{Meas}(X, S)$  is induced by an invertible measurable transformation between the underlying Borel spaces.

The primary tool in showing this is partial measurable line bundles. We begin by showing

**Theorem 19** *If  $F : \mathbf{Meas}(X, S) \rightarrow \mathbf{Meas}(Y, T)$  is an invertible additive functor with inverse  $\Phi : \mathbf{Meas}(Y, T) \rightarrow \mathbf{Meas}(X, S)$ , and  $\mathcal{H}$  is a partial measurable line bundle on  $X$ , the  $F(\mathcal{H})$  is a partial measurable line bundle on  $Y$ .*

**proof:** Suppose not. Now by Theorem 12,  $F(\mathcal{H})$  admits a direct summand  $\mathcal{L}$  which is a partial measurable line bundle with  $\text{supp}(\mathcal{L}) = \text{supp}(F(\mathcal{H}))$ . Since  $F(\mathcal{H})$  is not itself a partial measurable line bundle,  $\mathcal{L}^\perp$  has non-empty (measurable) support  $A \subset \text{supp}(\mathcal{H})$ , and there exists a partial measurable line bundle  $\mathcal{K} \subset \mathcal{L}^\perp$  with support  $A$ . Thus by Theorems 15 and 9 we have a decomposition of  $F(\mathcal{H})$  as

$$F(\mathcal{H}) \cong \mathcal{L}|_A \oplus \mathcal{L}|_{-A} \oplus \mathcal{K}|_A \oplus \mathcal{K}|_{-A} \oplus (\mathcal{L} + \mathcal{K})^\perp$$

(Note the last summand is the orthogonal complement of  $\mathcal{K}$  in  $\mathcal{L}^\perp$ .) But  $\mathcal{L}|_A \cong \mathcal{K}|_A$ , as they are both partial measurable line bundles with support  $A$ . Now applying  $\Phi$  to this decomposition gives

$$\mathcal{H} \cong \Phi(F(\mathcal{H})) \cong \Phi(\mathcal{L}|_A) \oplus \Phi(\mathcal{L}|_{-A}) \oplus \Phi(\mathcal{K}|_A) \oplus \Phi(\mathcal{K}|_{-A}) \oplus \Phi((\mathcal{L} + \mathcal{K})^\perp),$$

but since  $\Phi(\mathcal{L}|_A) \cong \Phi(\mathcal{K}|_A)$  contains a partial measurable line bundle with the same support, the fibers of  $\mathcal{H}$  at points of  $A$  have dimension greater than one, but this implies that  $\text{supp}(\Phi(\mathcal{L}|_A)) = \emptyset$ , i.e.  $\Phi(\mathcal{L}|_A) = 0$ . But then, we have  $\mathcal{L}|_A \cong F(\Phi(\mathcal{L}|_A)) \cong F(0) \cong 0$ , the last by the additivity of  $F$ . But this is a contradiction, since  $A \subset \text{supp}(\mathcal{H})$  was non-empty. •

We also have

**Proposition 20** *If  $F : \mathbf{Meas}(X, S) \rightarrow \mathbf{Meas}(Y, T)$  is an invertible additive functor with inverse  $\Phi : \mathbf{Meas}(Y, T) \rightarrow \mathbf{Meas}(X, S)$ , and  $\mathcal{H}$  and  $\mathcal{K}$  are partial measurable line bundles on  $X$ , then*

1.  $\text{supp}(\mathcal{H}) \subset \text{supp}(\mathcal{K})$  implies  $\text{supp}(F(\mathcal{H})) \subset \text{supp}(F(\mathcal{K}))$
2.  $\text{supp}(\mathcal{H}) \cap \text{supp}(\mathcal{K}) = \emptyset$  implies  $\text{supp}(F(\mathcal{H})) \cap \text{supp}(F(\mathcal{K})) = \emptyset$

**proof:** For the first statement, use Theorem 9 to decompose  $\mathcal{K}$  into a direct summand isomorphic to  $\mathcal{H}$  and a direct summand supported on  $\neg \text{supp}(\mathcal{H})$ . Apply  $F$ , and observe that the support of a direct summand is necessarily contained in the support of the direct sum. For the second statement, suppose not, decompose the direct sum of the images using Theorem 9 along the non-empty measurable set  $A = \text{supp}(F(\mathcal{H})) \cap \text{supp}(F(\mathcal{K}))$ . Taking images under  $\Phi$  yields the same sort of contradiction as in the proof of Theorem 19. •

In the case of invertible additive functors, this result essentially determines their structure:

**Corollary 21** *If  $F : \mathbf{Meas}(X, S) \rightarrow \mathbf{Meas}(Y, T)$  is an invertible additive functor, then  $F$  induces an isomorphism of  $\sigma$ -algebras by  $\text{supp}(\mathcal{L}) \mapsto \text{supp}(F(\mathcal{L}))$ , which is induced by pullback along an invertible measurable function  $\hat{F} : Y \rightarrow X$*

Of course, it is easy to establish the converse:

**Proposition 22** *If  $f : (Y, T) \rightarrow (X, S)$  is a measurable function with measurable inverse, then it induces an additive equivalence of categories between  $\mathbf{Meas}(X, S)$  and  $\mathbf{Meas}(Y, T)$  by pullback along  $f$  and its inverse.*

The two previous results taken together suggest, and almost suffice to show

**Theorem 23** *Any bounded additive isomorphism between categories of the form  $\mathbf{Meas}(X, S)$  which has a bounded inverse is naturally isomorphic to one induced by an inverse pair of measurable functions.*

**proof:** That the restriction of any such functor to the full subcategory of partial measurable line bundles both induces such a function on the underlying Borel spaces and is induced by one follows easily from Corollary 21 and Proposition 22.

Now, as any equivalence of categories preserves all limits and colimits which exist, it follows that the result holds if  $\mathbf{Meas}(X, S)$  is replaced by the full subcategory of all measurable fields of Hilbert spaces with a finite bound on the dimensions of their fibers, as these are all finite limits (or colimits) of partial measurable line bundles.

Thus the only real work, and the only place where the boundedness hypotheses are needed, is in constructing the components of the natural isomorphism at measurable fields of Hilbert spaces with unbounded (possibly infinite) fiber dimensions.

Let  $F : \mathbf{Meas}(X, S) \rightarrow \mathbf{Meas}(Y, T)$ ,  $F^{-1} : \mathbf{Meas}(Y, T) \rightarrow \mathbf{Meas}(X, S)$  be such an isomorphism. The restriction of  $F$  to partial measurable line bundles is induced by pullback along an isomorphism of Borel spaces  $\Phi$ .

We thus have another pair of inverse functors (easily seen to be  $*$ -functors and thus *a fortiori* bounded)  $\Phi^*$  and  $\Phi^{-1*}$ , and the restrictions of  $F$  and  $\Phi^*$  to the full subcategories of partial measurable line bundles, or to measurable fields of bounded dimension, coincide.

Let  $\mathcal{H}$  be a measurable field of Hilbert spaces on  $(X, S)$ . As every element of every fiber of  $\mathcal{H}$  is contained in a partial measurable line bundle whose inclusion may be taken to be norm preserving, and every pair of elements is contained in a subfield of Hilbert spaces with fiber dimensions less than or equal to two, it is easy to see that there is a linear isomorphism from  $F(\mathcal{H})_y$  to  $\Phi^*(\mathcal{H})_y$  for all  $y \in Y$ , induced by the linear isomorphisms between subfields with dimension bound two. We need to see that this family of linear isomorphisms, and the family of its inverses, in fact constitute a bounded field of bounded operators on  $(Y, T)$ .

Observe that  $\Phi^*$  is norm preserving—it is easy to see that pullbacks in general are  $*$ -functors and thus norm decreasing, but being invertible, it must preserve norms. So consider a vector  $\zeta_y \in \Phi^*(\mathcal{H})_y$ . It is a pullback of a vector  $\xi_x \in \mathcal{H}_x$  where  $\Phi(y) = x$ , which in turn lies in a section  $\xi$  which generates a subfield  $\iota_\xi : \langle \xi \rangle \rightarrow \mathcal{H}$ , where the inclusion  $\iota_\xi$  may be taken to be norm preserving. Applying  $F$  and  $\Phi^*$  to the inclusion gives inclusions  $F(\iota_\xi) : F(\langle \xi \rangle) \rightarrow F(\mathcal{H})$  and  $\Phi^*(\iota_\xi) : \Phi^*(\langle \xi \rangle) \rightarrow \Phi^*(\mathcal{H})$ .

As  $\Phi^*$  and  $\iota_\xi$  are norm preserving in the relevant senses,  $\Phi^*(\iota_\xi)$  is norm preserving. But  $F$  is bounded, say with bound  $N$ , and  $\iota_\xi$  is norm preserving and thus  $F(\iota_\xi)$  is bounded with bound  $N$ . But the linear isomorphism constructed above carries  $\zeta_y$  to the image of  $\zeta_y$  under  $F(\iota_\xi)$  (recall that  $F$  and  $\Phi^*$  coincide on partial measurable line bundles). Thus, the norm of  $\zeta_y$  is dilated by at most a factor of  $N$ . As this applies to any vector in any fiber, we have shown that the family of linear isomorphisms is a bounded family of bounded operators.

To see that the family of inverses is a bounded family of bounded operators, observe that an operator admits a bounded inverse exactly when its dilation of norms is bounded away from zero, say by  $1/N$ , and similarly that an invertible

functor between  $C^*$ -categories has a bounded inverse if and only if its dilations of norms are bounded away from 0. An identical argument using these lower bounds then suffices to show that the inverse family is a bounded family of operators.

Naturality of the family of bounded fields of operators thus constructed follows from the same considerations as linearity: it can be checked elementwise, and all elements lie in partial measurable line bundles. •

This last result also allows us to characterize the natural transformations between bounded invertible additive functors.

First, to describe natural endomorphisms of such functors, observe that by 1-composition with the inverse, it suffices to describe the natural endomorphisms of the identity functor:

**Theorem 24** *Any natural endomorphism of  $Id : \mathbf{Meas}(X, S) \rightarrow \mathbf{Meas}(X, S)$  is given by multiplication by a bounded measurable function  $\phi : X \rightarrow \mathbb{C}$ , and thus is bounded in the sense of Definition 18.*

**proof:** By construction in the proof of the previous theorem, any natural transformation is determined by its components at partial measurable line bundles. But by Theorem 9, these are determined by the component at the total measurable line bundle  $\mathbb{C}$ . Any map from  $\mathbb{C}$  to itself is given by multiplication by a bounded measurable function  $\phi$ , and it is easy to see that multiplication by any such function induces a natural endomorphism of  $Id$ . •

More generally the fact that the components on partial measurable line bundles determine natural transformations allows us to show

**Theorem 25** *Natural transformations from  $F : \mathbf{Meas}(X, S) \rightarrow \mathbf{Meas}(Y, T)$  to  $G : \mathbf{Meas}(X, S) \rightarrow \mathbf{Meas}(Y, T)$ , where both are invertible additive functors are given by bounded measurable functions  $\phi : E \rightarrow \mathbb{C}$ , where  $E$  is the equalizer of  $\widehat{F}^{-1}$  and  $\widehat{G}^{-1}$ .*

**proof:** Let  $\mathcal{H} = \mathbb{C}|_A$  be a partial measurable line bundle with support  $A$ . Then  $F(\mathcal{H})$  (resp.  $G(\mathcal{H})$ ) is a partial measurable line bundle on  $Y$  with support  $\widehat{F}^{-1}(A)$  (resp.  $\widehat{G}^{-1}(A)$ ). Thus maps between them are given by multiplication by measurable functions on  $\widehat{F}^{-1}(A) \cap \widehat{G}^{-1}(A)$  (and zero on other fibers).

Thus, in particular a natural transformation must induce (and be induced by) a family of measurable functions  $\phi_A : \widehat{F}^{-1}(A) \cap \widehat{G}^{-1}(A) \rightarrow \mathbb{C}$ . However, not every such family induces a natural transformation, as naturality squares impose consistency conditions. In particular, if we have a measurable set  $B \subset A$ , the direct sum decomposition  $\mathbb{C}|_A \cong \mathbb{C}|_B \oplus \mathbb{C}|_{A \setminus B}$  provides naturality squares for the inclusions and projections. From these it follows that  $\phi_A$  can be non-zero only on  $(\widehat{F}^{-1}(B) \cap \widehat{G}^{-1}(B)) \cup (\widehat{F}^{-1}(A \setminus B) \cap \widehat{G}^{-1}(A \setminus B))$ .

Letting  $B \subset A$  range over all containments of measurable subsets, we find, that  $\phi_A$  can only be non-zero on the points of  $A$  which are actually in the equalizer of  $\widehat{F}^{-1}$  and  $\widehat{G}^{-1}$ .

It is easy to see that multiplication by any such function induces a natural transformation. •

## 4 Direct Integrals

Classically the point of defining measurable fields of Hilbert spaces was to define a measure theoretic analogue of direct sums for the purpose of decomposing vonNeumann algebras and representations of non-compact groups.

In this section we investigate the functorial properties of this construction, and extend it to families of objects in  $\mathbf{Meas}(X, S)$ .

Recall

**Definition 26** *Given a (n almost) measurable field  $\mathcal{H}$  of Hilbert spaces on a Borel space  $(X, S)$  and a measure  $\mu$  on  $(X, S)$ , the direct integral*

$$\int_X^\oplus \mathcal{H}_x d\mu(x)$$

*is the Hilbert space of all measurable sections  $\xi \in \mathcal{M}_{\mathcal{H}}$  such that*

$$\|\xi\| = \left\{ \int_X \|\xi(x)\|^2 d\mu(x) \right\}^{\frac{1}{2}} < +\infty \quad (*)$$

*modulo the identification of measurable sections which are equal  $\mu$ -a.e. and equipped with the scalar product*

$$\langle \xi | \zeta \rangle = \int \langle \xi(x) | \zeta(x) \rangle d\mu(x)$$

*We call a section  $\xi$  satisfying  $(*)$  an  $L^2$  section of  $\mathcal{H}$ , and denote its  $\mu$ -a.e. equivalence class by  $\int_X^\oplus \xi(x) d\mu(x)$ .*

*Similarly given a ( $\mu$ -essentially) bounded field of operators  $\alpha : \mathcal{H} \rightarrow \mathcal{K}$ , the direct integral  $\int_X^\oplus \alpha(x) d\mu(x)$  is the map which takes an element  $\xi$  of  $\int_X^\oplus \mathcal{H}_x d\mu(x)$  to the element of  $\int_X^\oplus \mathcal{K}_x d\mu(x)$  given at each point  $x$  by  $\alpha(x)(\xi(x))$ .*

*For a ( $\mu$ -essentially) bounded field of operators  $\alpha(x)$ , we denote the map taking  $\int_X^\oplus \xi(x) d\mu(x)$  to the a.e.-equality class of the section  $x \mapsto \alpha(x)(\xi(x))$  by  $\int_X^\oplus \alpha(x) d\mu(x)$ .*

Now, it is easy to see that composition of bounded fields of operators is carried to composition of operators, and that the identity field is carried to the identity operator. Thus,  $\int_X^\oplus - d\mu(x)$  is a functor from  $\mathbf{Meas}(X, S)$  to  $\mathbf{Hilb}$  for any measure  $\mu$  on  $(X, S)$ .

It is also easy to see that two fields of operators which agree  $\mu$ -a.e. are mapped to the same operator between the direct integrals.

Several categorical properties of this construction will be important

**Theorem 27**  *$\int_X^\oplus - d\mu(x)$  is a  $\mathbb{C}$ -linear additive functor from  $\mathbf{Meas}(X, S)$  to  $\mathbf{Hilb} \cong \mathbf{Meas}(\{*\}, \{\{*\}, \emptyset\})$ . Moreover, if  $\mu$  is a probability measure, the functor  $\int_X^\oplus - d\mu(x)$  is bounded.*

**proof:**

It is immediate by construction that this construction preserves identity maps and carries the composition of bounded fields of operators to the composition of operators.

For  $\mathbb{C}$ -linearity, first note that once it is observed that the sum of two bounded fields of operators is a bounded field of operators, it is immediate that it acts on  $L^2$ -sections to give the sum of the action of its summands. Likewise, multiplication of all fibers by a complex scalar is a bounded field of operators, and induces the scalar multiplication on the a.e.-equality classes of  $L^2$  sections.

For additivity, note that the constant 0 field of Hilbert spaces is mapped to the Hilbert space 0. If we consider a biproduct in the category  $\mathbf{Meas}(X, S)$ , the equational conditions required for the direct integral to be a biproduct follow from functoriality and linearity.

Now let  $\alpha$  be a bounded field of operators. We then have

$$\left\{ \int_X \|\alpha(x)(\xi(x))\|^2 d\mu(x) \right\}^{\frac{1}{2}} \leq \sup_{x \in X} \|\alpha(x)\|_x \left\{ \int_X \|\xi(x)\|^2 d\mu(x) \right\}^{\frac{1}{2}}$$

But if  $\mu$  is a probability measure we have

$$\left\{ \int_X \|\xi(x)\|^2 d\mu(x) \right\}^{\frac{1}{2}} \leq \sup_{x \in X} \|\xi(x)\|_x = \|\xi\|$$

and thus  $\|\int_X^\oplus \alpha(x)(\xi(x)) d\mu(x)\| \leq \|\alpha\| \|\xi\|$ . Therefore  $\int_X^\oplus - d\mu(x)$  decreases the norm of arrows to which is applied, and is thus a bounded functor. •

One result which will be quite useful, as it will allow us to reduce most proofs to the case of probability measures is the following:

**Theorem 28** *If  $\mu \ll \nu$  then there is a natural transformation from  $\int_X^\oplus - d\nu(x)$  to  $\int_X^\oplus - d\mu(x)$  induced by multiplication by  $\sqrt{\frac{d\mu}{d\nu}}(x)$ , the square root of the Radon-Nikodym derivative.*

*If  $\mu \equiv \nu$ , then this natural transformation is a natural isomorphism with inverse induced by multiplication by  $\sqrt{\frac{d\nu}{d\mu}}(x)$ .*

**proof:** Recall that the Radon-Nikodym derivative is always a measurable function, and that square-root of non-negative functions preserves measurability. Thus multiplication by  $\sqrt{\frac{d\mu}{d\nu}}(x)$  takes measurable vector fields for  $\mathcal{H}$  to measurable vector fields by Lemma 13.

Thus the map on measurable fields of Hilbert spaces takes a measurable vector field  $\xi(x)$  to  $\sqrt{\frac{d\mu}{d\nu}}(x)\xi(x)$ .

But

$$\int \left\| \sqrt{\frac{d\mu}{d\nu}}(x) \xi(x) \right\|_x^2 d\nu(x) = \int \|\xi(x)\|^2 \frac{d\mu}{d\nu}(x) d\nu(x) = \int \|\xi(x)\|_x^2 d\mu(x)$$

and thus the map induces a map on direct integrals as desired.

Naturality follows from the fact that the map is induced by a central endomorphism of the measurable field of Hilbert spaces.

The second statement follows from the chain rule for Radon-Nikodym derivatives. •

Since any  $\sigma$ -finite measure is equivalent to a probability measure, we have

**Corollary 29** *Any direct integral functor  $\int_X^\oplus (-) d\mu(x)$  is naturally equivalent to a direct integral functor  $\int_X^\oplus (-) d\nu(x)$  for a probability measure  $\nu$  on  $X$ .*

We can generalize the direct integral construction to give rise to an important family of functors between categories of the form  $\mathbf{Meas}(X, S)$ . In fact, these functors will allow us to decompose many objects in categories of measurable fields as direct integrals of simpler objects.

**Definition 30** *For a measurable function  $\Phi : (X, S) \rightarrow (Y, T)$  between Borel spaces, a  $\Phi$ -fibred measure on  $X$  is a uniformly totally  $\sigma$ -finite conditional measure distribution  $\mu_y$ , that is a  $Y$ -indexed family of measures on  $X$  such that*

1.  $\mu_y(X \setminus \Phi^{-1}(y)) = 0$
2. For all  $A \in S$  the function  $y \mapsto \mu_y(A)$  is measurable
3. There exist a sequence of measurable sets  $A_n$  such that  $X = \cup_n A_n$  and for all  $y \in Y$  and all  $n$   $\mu_y(A_n) < \infty$

We then have

**Proposition 31** *If  $f : X \rightarrow \mathbb{R}$  (or  $\mathbb{C}$ ) is measurable and  $\mu_y$  is a  $\Phi$ -fibred measure on  $X$  for a measurable function  $\Phi : X \rightarrow Y$ , then the function  $y \mapsto \int f d\mu_y(x)$  is a measurable function on  $Y$*

**proof:** It suffices to consider the case of  $f$  real-valued and non-negative. Now, let  $f_n$  be a sequence of simple functions approximating  $f$  from below.

The functions  $y \mapsto \int f_n d\mu_y(x)$  are all measurable, being real linear combinations of functions of the form  $y \mapsto \mu_y(A)$  (for the  $A$ 's on which  $f_n$  is constant). But  $[y \mapsto \int f d\mu_y(x)] = \limsup [y \mapsto \int f_n d\mu_y(x)]$ , and thus is measurable, as the lim sup of a sequence of measurable functions is again measurable. •

This last result is useful to us because in generalizing direct integrals to give functors between categories of the form  $\mathbf{Meas}(X, S)$ . It will let us show that our constructions preserve measurable sections.

We also have a fibred analog of our earlier reduction to probability measures:

**Theorem 32** *If  $\mu_y$  is a  $\Phi$ -fibred measure for a measurable  $\Phi : (X, S) \rightarrow (Y, T)$ , there exists a  $\Phi$ -fibred measure  $\nu_y$  such that for all  $y$   $\mu_y \equiv \nu_y$  and each  $\nu_y$  is a probability measure on  $X$ .*

**proof:** The only catch in simply applying the standard construction which shows that any  $\sigma$ -finite measure is equivalent to a probability measure in each fiber separately is the need to ensure that all the maps  $y \mapsto \nu_y(A)$  for  $A \in S$  will be measurable.

Let  $A_n$  be a sequence of disjoint measurable sets in  $X$  such that  $X = \cup_{n=1}^{\infty} A_n$  and all the  $\mu_y(A_n)$ 's are finite. Define  $h : X \times Y \rightarrow \mathbb{R}$  by

$$h(x, y) = \{2^n \mu_y(A_n)\}^{-1} \quad x \in A_n.$$

We can then let  $\nu_y(A) = \int_A h(x, y) d\mu_y(x) = \sum_{n=1}^{\infty} \{2^n \mu_y(A_n)\}^{-1} \mu_y(A_n \cap A)$ .

But this is the lim sup of its partial sums, which are measurable since addition, multiplication, multiplication by constants, and reciprocation all preserve measurability, and the functions  $y \mapsto \mu_y(A_n)$  and  $y \mapsto \mu_y(A_n \cap A)$  are both measurable. Thus we are done. •

We can then make

**Definition 33** Let  $\Phi : (X, S) \rightarrow (Y, T)$  be a measurable function between Borel spaces. Let  $\mu_y(x)$  be a  $\Phi$ -fibered measure on  $X$ .

For a(n almost) measurable field of Hilbert spaces  $\mathcal{H}$  on  $(X, S)$ , let

$$\int_{\Phi}^{\oplus} \mathcal{H}_x d\mu_y(x)$$

denote the (almost) measurable field of Hilbert spaces with fiber at  $y$  given by  $\int_X^{\oplus} \mathcal{H}_x d\mu_y(x)$ , and measurable sections given by the closure under condition 2 of the image of the set of measurable sections of  $\mathcal{H}$ .

Observe that this definition makes sense, since the fiberwise scalar product on  $\int_{\Phi}^{\oplus} \mathcal{H}_x d\mu_y(x)$  is given by

$$\left\langle \int_X^{\oplus} \xi(x) d\mu_y(x) \mid \int_X^{\oplus} \zeta(x) d\mu_y(x) \right\rangle_y = \int \langle \xi(x) \mid \zeta(x) \rangle d\mu_y(x)$$

which is the fiberwise integral of a measurable function whenever  $\xi$  and  $\zeta$  are measurable sections of a(n almost) measurable field of Hilbert spaces, and thus is measurable by the definition of a  $\Phi$ -fibered measure.

In the measurable case the resulting field of Hilbert spaces admits a fundamental sequence by virtue of the following two lemmas:

**Lemma 34** If  $\mu$  (resp.  $\mu_y$ ) is a  $\sigma$ -finite measure (resp.  $\Phi$ -fibered measure) on  $X$ , then any measurable field of Hilbert spaces on  $X$  admits a fundamental sequence of  $L^2$ -sections with respect to  $\mu$  (resp. of sections which are  $L^2$ -sections with respect to all of the  $\mu_y$ 's).

**proof:** Let  $\{A_n\}_{n=1}^{\infty}$  be the sequence of measurable sets exhausting  $X$  of finite measure with respect to  $\mu$  (resp. with respect to all of the  $\mu_y$ ). Let  $\{\xi_i\}_{i=1}^{\infty}$  be a fundamental sequence for  $\mathcal{H}$ . Let



$$\zeta_{i,n,\alpha}(x) = \frac{\alpha}{\|\xi_i(x)\|} \xi_i|_{A_n}(x)$$

for  $i, n = 1, 2, \dots$  and  $\alpha$  any algebraic number. Plainly the  $\zeta_{i,n,\alpha}(x)$  are dense in each  $\mathcal{H}_x$ . But they are also  $L^2$  sections with respect to  $\mu$  (resp. with respect to all  $\mu_y$ ), since their norms are easily seen to be given by

$$\|\zeta_{i,n,\alpha}\| = \alpha \sqrt{\mu(A_n)} < +\infty$$

(resp. the same *mutatis mutandis* for each  $\mu_y$ ). Now, for technical reasons we will replace this sequence with the sequence of all finite sums of its elements. •

**Lemma 35** *If  $L^2(X, \mu)$  is separable (resp.  $L^2(X, \mu_y)$  forms a measurable field of Hilbert spaces on  $Y$ ), then for any measurable field of Hilbert spaces  $\mathcal{H}_x$  on  $X$  which admits a fundamental sequence of  $L^2$ -sections with respect to  $\mu$  (resp. of sections which are  $L^2$  with respect to all of the  $\mu_y$ ), the direct integral  $\int_X^\oplus \mathcal{H}_x d\mu(x)$  is a separable Hilbert space (resp. the almost measurable field of Hilbert spaces  $\int_X^\oplus \mathcal{H}_x d\mu_y(x)$  on  $Y$  is a measurable field of Hilbert spaces.)*

**proof:** First recall that the existence of a countable dense set in a Hilbert space is equivalent to the existence of countable orthonormal basis. A similar argument establishes that the existence of a fundamental sequence in an almost measurable field of Hilbert spaces (i.e. the measurability of the field of Hilbert spaces) is equivalent to the existence of a countable set of sections which are orthogonal in each fiber, whose norms in all fibers is either 1 or 0. (Apply Gram-Schmidt fiberwise.) Call such a countable set of sections a foundation.<sup>3</sup>

Consider  $\{\phi_j\}$ , an orthonormal basis for  $L^2(X, \mu)$  (resp. a foundation for  $L^2(X, \mu_y)$ ), and  $\{\xi_i\}$ , a foundation for  $\mathcal{H}$ . We claim the set  $\{\int_X^\oplus \phi_j \xi_i d\mu\}$  (resp.  $\{\int_X^\oplus \phi_j \xi_i d\mu_y\}$ ) is a total set in  $\int_X^\oplus \mathcal{H}_x d\mu(x)$  (resp. is total in  $\int_X^\oplus \mathcal{H}_x d\mu_y(x)$  for all  $y$ ), and thus its algebraic span is dense (resp. dense in each fiber).

Now, observe that  $\|\phi_j(x)\xi_i(x)\|_x^2 = |\phi_j(x)|^2 \|x i_i(x)\|_x^2 \leq |\phi_j|^2$ . Since  $\phi_j$  is  $L^2$  it is certainly measurable, and thus  $\phi_j \xi_i$  is a measurable sect of its fiber-wise norm is majorized by the absolute square of the  $L^2$  function  $\phi_j$ . It thus follows that  $\phi_j \xi_i$  is an  $L^2$ -section.

It suffices to handle the case of a single measure space  $(X, \mu)$ . Suppose  $\int_X^\oplus \zeta d\mu$  is an element of  $\{\int_X^\oplus \phi_j \xi_i d\mu\}$ , and that  $\langle \int_X^\oplus \zeta d\mu | \int_X^\oplus \phi_j \xi_i d\mu \rangle = 0$  for all  $i$  and  $j$ . But pulling out the  $\phi_j$  by sequilinearity, we find that  $\langle \zeta | \xi_i \rangle$  is orthogonal to all of the  $\phi_j$ 's, and thus is zero in  $L^2(X, \mu)$ , since the  $\phi_j$ 's are total.<sup>4</sup> Thus  $\langle \zeta | \xi_i \rangle$  is zero  $\mu$ -a.e., and thus  $\zeta$  is zero  $\mu$ -a.e. since the  $x i_i$ 's are total in each fiber. Thus  $\zeta$  is zero in the direct integral, and we are done. •

<sup>3</sup>We want a word other than basis since it need not be a basis in each fiber, and may not even be linearly independent globally.

<sup>4</sup>As an aside, notice that this argument, together with the classical proof of duality between  $L^p$  spaces shows that when ever  $\zeta$  and  $\xi$  are  $L^2$  sections of a measurable field of Hilbert spaces, the function  $x \mapsto \langle \zeta(x) | \xi(x) \rangle_x$  is an  $L^2$  function.

At this point, we should point out why the preservation of measurability under fiberwise integration is a non-vacuous condition, and provide some useful examples of fibered measures.

We begin with an example of a family of measures on the inverse images under a measurable map which is not a fibered measure:

**Example 36** Consider the second projection map  $p_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Now, let  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  be a non-negative non-measurable function. Consider then the family of measures  $\mu_y$  given by Borel measure in the first coördinate  $x$  on the subsets of  $[0, \beta(y)] \times \{y\} \subset p_2^{-1}(y) \subset \mathbb{R}^2$ , with sets lying in  $p_2^{-1} \setminus [0, \beta(y)] \times \{y\}$  having measure zero.

The fiberwise integral of the (obviously measurable) constant function 1 on  $\mathbb{R}^2$ , which could arise, for instance, as the scalar product of the constant section 1 in the constant field of Hilbert spaces  $\mathbb{C}$  on  $\mathbb{R}^2$  with itself, is then the non-measurable function  $\beta$ .

However, it is easy to give examples of fibered measures:

The following is essentially Fubini's Theorem for measurability:

**Example 37** Let  $(X, S)$  and  $(Y, S)$  be any Borel spaces, and let  $\mu$  be a measure on  $X$ , the family of measures given by  $\mu_y(A) = \mu(p_1(A \cap (X \times y)))$  is a  $p_2$ -fibered measure. Condition 1 is true by construction, condition 2 is immediate since the functions  $y \mapsto \mu_y(A)$  are all constant functions, while for condition 3, if  $A_n \subset X$  are a sequence of sets which witnesses to the  $\sigma$ -finiteness of  $\mu$ , then  $A_n \times X$  provide the necessary sequence for the uniform  $\sigma$ -finiteness of  $\mu_y$ .

Another example, which is the simplest example of a useful family we will introduce later, is:

**Example 38** Let  $(X, S)$  be any Borel space. Given  $p_2 : X \times X \rightarrow X$ , the family of measures  $\mu_y$  on  $X \times X$  given by

$$\mu_y(A) = \begin{cases} 1 & \text{if } (y, y) \in A \\ 0 & \text{otherwise} \end{cases}$$

Again condition 1 is immediate, while any countable sequence of measurable sets exhausting  $X \times X$  will suffice for condition 3. For condition 2, observe that the projection functions are measurable, the diagonal  $\Delta$  is measurable and the function  $y \mapsto \mu_y(A)$  in this case is simply the characteristic function of  $\Delta \cap A$ .

It is easy to see that this family is a fibered measure: integration with respect to it is simply restriction to the diagonal, which is a measurable subset of  $(X, S)$ .

Similarly, given any section  $s$  of the projection with a measurable image, the family of measures concentrated on  $(s(y), y)$  giving each of these points measure 1 in its fiber is a fibered measure.

$\Phi$ -fibered measures are closed under multiplication by measurable functions on the target, under addition, and suitable adaptations of limiting processes which preserve measurable functions.

## 5 Measurable Functors

Having seen that  $\mathbb{C}$ -linear invertible additive functors are induced by invertible measurable functions between the underlying Borel space, we now turn to the question of what class of functors including these are most appropriate to consider when forming a 2-category of categories of (almost) measurable fields of Hilbert spaces.

One obvious approach is to consider those  $\mathbb{C}$ -linear functors which respect the norm structure in the sense that the map induced on hom-sets is fiberwise continuous with respect to the operator norm.

We will prefer a structural rather than an axiomatic approach—though we conjecture that the class of functors just described is in fact the same as that we are about to define.

Given an almost measurable field of Hilbert spaces  $\mathcal{K}$  on  $(X \times Y, S \star T)$ , and a  $p_2$ -fibred measure  $\mu_y$  on  $X \times Y$ , we may construct a  $\mathbb{C}$ -linear additive functor  $\Phi_{\mathcal{K}, \mu_y}$ :

Let

$$\Phi_{\mathcal{K}, \mu_y}(\mathcal{H})_y = \int_X^{\oplus} \mathcal{H}_x \otimes \mathcal{K}_{<x, y>} d\mu_y(x)$$

with  $\mathcal{M}_{\Phi(\mathcal{H})}$  given as the closure under condition 2 of the set

$$\left\{ \int_X^{\oplus} \eta(x) \otimes \kappa(x, y) d\mu_y(x) \mid \eta \in \mathcal{M}_{\mathcal{H}}; \kappa \in \mathcal{M}_{\mathcal{K}} \right\}.$$

Observe that there is a subtlety here: not all pairs of sections  $\eta, \kappa$  will define a section, only those which, for every  $y \in Y$  give rise to  $L^2$ -sections of the tensor product.

**Definition 39** *A functor from  $\mathbf{Meas}(X, S)$  to  $\mathbf{Meas}(Y, T)$  is measurable if it is  $\mathbb{C}$ -linear equivalent to one of the form  $\Phi_{\mathcal{K}, \mu_y}$ .*

We can then show the functors we considered earlier all belong to this new class:

**Theorem 40** *Any invertible additive functor between categories of the form  $\mathbf{Meas}(X, S)$  is a measurable functor.*

**proof:** Let  $F : \mathbf{Meas}(X, S) \rightarrow \mathbf{Meas}(Y, T)$  be an invertible additive functor. By Corollary 21 it is induced by pullback along an invertible measurable function  $\widehat{F} : (Y, T) \rightarrow (X, S)$ . If we then consider the constant field of Hilbert spaces  $\mathbb{C}$  on  $X \times Y$  and the  $p_2$ -fibred measure

$$\mu_y(A) = \begin{cases} 1 & \text{if } (\widehat{F}(y), y) \in A \\ 0 & \text{otherwise} \end{cases},$$

it is easy to see that the pullback functor is naturally isomorphic to the functor  $\Phi_{\mathbb{C}, \mu_y} \cdot \bullet$ .

**Theorem 41** *If  $\phi : (X, S) \rightarrow (Y, T)$  is a measurable function between Borel spaces and  $\mu_y$  is a  $\phi$ -fibred measure on  $X$ , then the functor  $\int_\phi^\oplus (-) d\mu_y(x)$  is a measurable functor.*

**proof:** Define a  $p_2$ -fibred measure on  $X \times Y$  by

$$\tilde{\mu}_y(A) = \mu_y(\{x | (x, \phi(x)) \in A\})$$

It is easy to see that  $\int_\phi^\oplus (-) d\mu_y(x)$  is naturally isomorphic to  $\Phi_{\mathbb{C}, \tilde{\mu}_y}$ , and thus measurable. •

One vexing thing about measurable functors is the fact that it is not immediate that the composition of measurable functors is measurable.

To show this, we will need to invoke Maharam's result [8] on disintegration of measures, and several results relating tensor products and direct integrals.

**Proposition 42** *If  $\mathcal{H}$  is a measurable field of Hilbert spaces on  $X$  and  $\mathcal{K}$  is a separable Hilbert space, then there is a measurable field of Hilbert spaces  $\mathcal{H} \otimes \mathcal{K}$  with fiber at  $x$  given by  $\mathcal{H}_x \otimes \mathcal{K}$ , with all algebraic-coefficient finite linear combinations of tensor products of elements in the fundamental sequence of  $\mathcal{H}$  with elements of a countable dense set in  $\mathcal{K}$  as fundamental sequence, and the closure under condition 2 of this sequence as  $\mathcal{M}_{\mathcal{H} \otimes \mathcal{K}}$ .*

**proof:** By [9] IV §8 Lemma 8.10, it suffices to show that the proposed fundamental sequence is fiberwise dense and that all of the pairwise scalar product functions are measurable. Density is clear from the construction of Hilbert space tensor products.

For the other condition observe that

$$\begin{aligned} & \langle \sum_{i,j} a_{i,j} \xi_j(x) \otimes \zeta_i | \sum_{k,l} b_{k,l} \phi_l(x) \otimes \omega_k \rangle_x = \\ & \sum_{i,j,k,l} a_{i,j} \overline{b_{k,l}} \langle \zeta_i | \omega_k \rangle_{\mathcal{K}} \langle \xi_j(x) | \phi_l(x) \rangle_x, \end{aligned}$$

is measurable as a function of  $x$  since all of the  $x \mapsto \langle \xi_j(x) | \phi_l(x) \rangle_x$  are measurable, and linear combinations of measurable functions are measurable (all other expressions occurring in the last are constant in  $x$ ). •

Moreover we have

**Theorem 43** *There is a canonical natural isomorphism*

$$\int_X^\oplus \mathcal{H} \otimes \mathcal{K}_x d\mu(x) \cong \left( \int_X^\oplus \mathcal{H}_x d\mu(x) \right) \otimes \mathcal{K}$$

where  $\mathcal{K}$  is any separable Hilbert space and  $\mathcal{H}$  is any measurable field of Hilbert spaces on a Borel space  $X$ , and  $\mathcal{H} \otimes \mathcal{K}$  is as in the previous proposition.

**proof:** By Corollary 29 we may assume without loss of generality that  $\mu$  is a probability measure.

We proceed by first constructing a canonical map from  $\int_X^\oplus \mathcal{H}_x d\mu(x) \otimes \mathcal{K}$  to  $\int_X^\oplus \mathcal{H} \otimes \mathcal{K}_x d\mu(x)$ , then showing that it is an isomorphism and natural in both variables.

Now, since any element in  $\int_X^\oplus \mathcal{H}_x d\mu(x) \otimes \mathcal{K}$  is the limit of a Cauchy sequence of elements in the algebraic tensor product, to specify a bounded linear operator from  $\int_X^\oplus \mathcal{H}_x d\mu(x) \otimes \mathcal{K}$  to any other Hilbert space, it suffices to specify its behavior on elements of the form  $\int^\oplus \xi(x) d\mu(x) \otimes \zeta$  for  $\zeta \in \mathcal{K}$  and  $\xi(x)$  a  $L^2$ -section of  $\mathcal{H}$ .

Rather obviously we wish to map  $\int^\oplus \xi(x) d\mu(x) \otimes \zeta$  to  $\int^\oplus \xi(x) \otimes \zeta d\mu(x)$ .

To see that this in fact defines a bounded operator, we will need to verify that

1.  $\xi(x) \otimes \zeta$  is a measurable vector field in  $\mathcal{H} \otimes \mathcal{K}$ ,
2.  $\left\{ \int \|\xi(x) \otimes \zeta\|_x^2 d\mu(x) \right\}^{\frac{1}{2}} < \infty$ ,
3. there exists an  $M$  such that for all  $\xi(x)$  and  $\zeta$

$$\left\{ \int \|\xi(x) \otimes \zeta\|_x^2 d\mu(x) \right\}^{\frac{1}{2}} \leq M \left\{ \int \|\xi(x)\|_x^2 d\mu(x) \right\}^{\frac{1}{2}} \|\zeta\|_{\mathcal{K}}$$

and

4. the image of each element is independent of the choice of  $\mu$ -a.e. equality class representative  $\xi(x)$ .

For the first consider an element of the fundamental sequence,  $\sum_{i,j} a_{i,j} \xi_i(x) \otimes \zeta_j$ , and form the function of  $x$  given by scalar product with  $\xi(x) \otimes \zeta$ . By sequilinearity this reduces to a linear combination of the functions  $x \mapsto \langle \xi_i(x) | \xi(x) \rangle$ , and is thus measurable.

For the second and third, we compute

$$\begin{aligned} \left\{ \int \|\xi(x) \otimes \zeta\|^2 d\mu(x) \right\}^{\frac{1}{2}} &= \left\{ \int \|\xi(x)\|_x^2 \|\zeta\|_{\mathcal{K}}^2 d\mu(x) \right\}^{\frac{1}{2}} \\ &= \|\zeta\| \left\{ \int \|\xi(x)\|^2 d\mu(x) \right\}^{\frac{1}{2}} \end{aligned}$$

This is finite since the integral in the last right-hand side is the norm of the  $L^2$ -section  $\xi(x)$  of  $\mathcal{H}$ , that is the norm in the direct integral  $\int^\oplus \mathcal{H}_x d\mu(x)$ . Moreover, notice that the right-hand side as a whole is the norm of the preimage  $\int \xi(x) d\mu(x) \otimes \zeta$ , and thus  $M = 1$  suffices.

For the independence of the choice of  $\mu$ -a.e. equality representative, observe that if  $\xi = \xi'$   $\mu$ -a.e., then  $\xi \otimes \zeta = \xi' \otimes \zeta$   $\mu$ -a.e.

We have already shown more than that this map simply exists. In showing that it was bounded, we actually showed that it preserved the norm on a dense set, and thus is an isometry onto its image.

It thus remains only to show that it is surjective. But, by completeness, it suffices to show that its image is dense in  $\int^{\oplus} \mathcal{H} \otimes \mathcal{K} d\mu(x)$ .

Carrying out the proof of [9] IV §8 Lemma 8.12 (simultaneous fiberwise Gram-Schmidt orthonormalization) with the fundamental sequence for  $\mathcal{H} \otimes \mathcal{K}$  gives the same result with one added feature:

**Lemma 44** *There is a sequence of measurable vectorfields  $\psi_i$  in  $\mathcal{H} \otimes \mathcal{K}$  such that*

1.  $\{\psi_i(x) | 1 \leq i \leq \dim \mathcal{H}_x\}$  is an orthonormal basis for  $\mathcal{H}_x$ ,
2. for  $i > \dim \mathcal{H}_x$   $\psi_i(x) = 0$
3.  $\psi_i = \sum_{k,l} a_{k,l}^i \xi_l \otimes \zeta_k$  for a finite set of measurable functions  $a_{k,l}^i(x)$ , that is,  $\psi_i$  lies in the  $Meas(X)$ -linear span of the fundamental sequence of  $\mathcal{H} \otimes \mathcal{K}$ .

Observe that the fundamental sequence of  $\mathcal{H} \otimes \mathcal{K}$  lies in the image of the map. It also follows from the bilinearity of  $\otimes$  and Lemma 13 that all elements of the  $Meas(X)$ -linear span are in the image of the map.

We can thus form a sequence of measurable subfields

$$\mathcal{L}^n = Span_{Meas(X)} \{\psi_i | i = 1, \dots, n\},$$

and fields of bounded operators  $p_n$  and  $p_n^\perp$  projecting onto these and their orthogonal complements. Moreover each of the  $\mathcal{L}^n$  lies in the image of the map.

Now, observe that  $\psi = p_n(\psi) + p_n^\perp(\psi)$ , and that the summands are orthogonal in each fiber.

It thus follows that

$$\|p_n(\psi)(x)\|^2 \leq \|\psi(x)\|^2$$

and

$$\|p_n^\perp(\psi)(x)\|^2 \leq \|\psi(x)\|^2$$

By Proposition 42  $x \mapsto \|p_n(\psi)(x)\|^2$  is measurable, and since it is majorized by the integrable function  $x \mapsto \|\psi(x)\|^2$  it is integrable. Thus  $x \mapsto \|p_n^\perp(\psi)(x)\|^2$  is also integrable, being the difference of two integrable functions.

Each of  $p_n(\psi)$  and  $p_n^\perp(\psi)$  thus represents an element in  $\int^{\oplus} \mathcal{H}_x \otimes \mathcal{K} d\mu(x)$ , and as noted above the  $p_n(\psi)$  lie in the image of the map.

It thus suffices to show that the sequence  $\{p_n(\psi)\}$  converge to  $\psi$ , not merely pointwise in each fiber, but with respect to the norm on the direct integral.

For any  $\epsilon > 0$  let  $E_n^\epsilon = \{x \mid \|\psi(x) - p_n(\psi)(x)\|_x^2 = \|p_n^\perp(\psi)(x)\|_x^2 < \epsilon\}$ .

Observe that

1.  $m > n$  implies  $E_m^\epsilon \supseteq E_n^\epsilon$ ;

2. all of the  $E_n^\epsilon$  are measurable (being unions of sets  $N(f) \cap f^{-1}((-\infty, \epsilon))$  and  $X \setminus (N(f) \cap f^{-1}(\mathbb{R}))$ , for  $f(x) = \|p_n^\perp(\psi)(x)\|_x^2$ ); and
3.  $\bigcup_{n=1}^\infty E_n^\epsilon = X$ , since for all  $x$   $\lim_{n \rightarrow \infty} p_n(\psi)(x) = \psi(x)$ .

Thus we also have for any  $\epsilon > 0$  that

$\int_{E_n^\epsilon} \|\psi(x)\|^2 d\mu(x)$  is an increasing sequence with limit  $M = \int_X \|\psi(x)\|^2 d\mu(x)$ .  
Now, fix  $\epsilon > 0$ . By the foregoing discussion, we can choose an  $N$  such that

$$M - \int_{E_N^{\frac{\epsilon}{2}}} \|\psi(x)\|^2 d\mu(x) < \frac{\epsilon}{2}.$$

For any  $n \geq N$ , we then have

$$\begin{aligned} \|\psi - p_n(\psi)\|_{\mathcal{H} \otimes \mathcal{K}}^2 &= \int_X \|\psi(x) - p_n(\psi)(x)\|_x^2 d\mu(x) \\ &= \int_{E_N^{\frac{\epsilon}{2}}} \|\psi(x) - p_n(\psi)(x)\|_x^2 d\mu(x) + \\ &\quad \int_{X \setminus E_N^{\frac{\epsilon}{2}}} \|\psi(x) - p_n(\psi)(x)\|_x^2 d\mu(x). \end{aligned}$$

But

$$\begin{aligned} \int_{E_N^{\frac{\epsilon}{2}}} \|\psi(x) - p_n(\psi)(x)\|_x^2 d\mu(x) &< \int_{E_N^{\frac{\epsilon}{2}}} \frac{\epsilon}{2} d\mu(x) \\ &\leq \frac{\epsilon}{2}. \end{aligned}$$

the first inequality by the construction of  $E_N^{\frac{\epsilon}{2}}$ , and the fact that  $E_N^{\frac{\epsilon}{2}} \subset E_n^{\frac{\epsilon}{2}}$ , the second because we are integrating over a probability space, while

$$\begin{aligned} \int_{X \setminus E_N^{\frac{\epsilon}{2}}} \|\psi(x) - p_n(\psi)(x)\|_x^2 d\mu(x) &= \int_{X \setminus E_N^{\frac{\epsilon}{2}}} \|p_n^\perp(\psi)(x)\|_x^2 d\mu(x) \\ &\leq \int_{X \setminus E_N^{\frac{\epsilon}{2}}} \|\psi(x)\|_x^2 d\mu(x) \\ &= M - \int_{E_N^{\frac{\epsilon}{2}}} \|\psi(x)\|^2 d\mu(x) \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Thus the image is dense, and the map is an isomorphism.

Naturality in both variables follows easily by chasing the images of elements of the form  $\int^\oplus \xi(x) d\mu(x) \otimes \zeta$ . •

**Theorem 45** *The composition of two measurable functors is a measurable functor.*

**proof:** Consider the functors  $F = \Phi_{\mathcal{F}, \mu_y}$  and  $G = \Phi_{\mathcal{G}, \nu_z}$ , for  $\mathcal{F}$  (resp.  $\mathcal{G}$ ) a measurable field of Hilbert spaces on  $X \times Y$  (resp.  $Y \times Z$ ) and  $\mu_y$  (resp.  $\nu_z$ ) a  $p_2$ -fibered measure on  $X \times Y$  (resp.  $Y \times Z$ ). Without loss of generality we may assume that all of the  $\mu_y$  and  $\nu_z$  are probability measures.

We then have

$$G(F(\mathcal{H}))_z = \int_Y^\oplus \left\{ \int_X^\oplus \mathcal{H}_x \otimes \mathcal{F}_{(x,y)} d\mu_y(x) \right\} \otimes \mathcal{G}_{(y,z)} d\nu_z(y)$$

By the functoriality of the outer direct integral and the previous theorem, this is then naturally isomorphic to

$$\int_Y^\oplus \int_X^\oplus \mathcal{H}_x \otimes \mathcal{F}_{(x,y)} \otimes \mathcal{G}_{(y,z)} d\mu_y(x) d\nu_z(y)$$

Now, for each  $z \in Z$   $\int \mu_y d\nu_z(y)$  is a probability measure on  $X \times Y$ . Define a  $z$ -indexed family of measures on  $X$  by  $\lambda_z(A) = \int \mu_y d\nu_z(y)(A \times Y)$ . By abuse of notation, we also denote by  $\lambda_z$  the family of measures on  $X \times Z$  given by  $\lambda_z(B) = \lambda_z(p_1(B \cap (X \times \{z\})))$ , where  $\lambda_z$  on the right-hand side is the measure just defined.

**Lemma 46** *The family of measures  $\lambda_z$  on  $X \times Z$  is a  $p_2$ -fibered measure.*

**proof:** By construction the first condition of Definition 30 is satisfied. The second follows from the corresponding condition for  $\mu_y$  and  $\nu_z$ , while the third is immediate once it is observed that the  $\lambda_z$  are all probability measures. •

Observe also that by construction  $\int \mu_y d\nu_z(y)$  satisfies the hypotheses of the following theorem of Maharam [8] (cf. also [6]) with respect to the projection  $p_1$  onto  $X$  and each of the  $\lambda_z$ :

**Theorem 47** (Maharam) *Let  $L$  be a Lusin space and  $S$  a non-empty Suslin space,  $p$  a measurable map from  $L$  to  $S$  each equipped with the usual Borel structure. A  $\sigma$ -finite measure  $\mu$  on the Borel sets of  $L$  has a uniformly  $\sigma$ -finite disintegration with respect to  $p$  and  $\nu$  a measure on  $S$  if and only for all measurable  $B$  in  $S$   $\nu(B) = 0$  implies  $\mu(p^{-1}(B)) = 0$*

We thus have a  $Z$ -indexed family of  $p_1$ -fibered measures  $\kappa_{x,z}$  such that

$$\int \kappa_{x,z} d\lambda_z(x) = \int \mu_y d\nu_z(y)$$

Thus the object

$$\int_Y^\oplus \int_X^\oplus \mathcal{H}_x \otimes \mathcal{F}_{(x,y)} \otimes \mathcal{G}_{(y,z)} d\mu_y(x) d\nu_z(y)$$



can also be described as

$$\int_Y^\oplus \int_X^\oplus \mathcal{H}_x \otimes \mathcal{F}_{(x,y)} \otimes \mathcal{G}_{(y,z)} d\kappa_{x,z}(y) d\lambda_z(x)$$

Thus, applying the previous theorem and the functoriality of the outer direct integral, we see that this is naturally isomorphic to

$$\int_Y^\oplus \mathcal{H}_x \otimes \left[ \int_X^\oplus \mathcal{F}_{(x,y)} \otimes \mathcal{G}_{(y,z)} d\kappa_{x,z} \right] d\lambda_z(x).$$

Thus the composition is measurable, being naturally isomorphic to the functor  $\Phi_{\mathcal{K}, \lambda_z}$  where

$$\mathcal{K}_{(x,z)} = \int_Y^\oplus \mathcal{F}_{(x,y)} \otimes \mathcal{G}_{(y,z)} d\kappa_{x,z} = \int_{p_{1,3}}^\oplus p_{1,2}^* \mathcal{F}_{(x,y)} \otimes p_{2,3}^* \mathcal{G}_{(y,z)} d\kappa_{x,z}$$

is a measurable field by the construction of Definition 33. •

In the same way, we can isolate an interesting class of natural transformations between measurable functors:

Consider two parallel measurable functors

$$\Phi_{\mathcal{F}, \mu_y}, \Phi_{\mathcal{G}, \nu_y} : \mathbf{Meas}(X, S) \rightarrow \mathbf{Meas}(Y, T).$$

Now if the fibered measures  $\mu_y$  and  $\nu_y$  are equal, it is clear that any measurable field of operators on  $X \times Y$  from  $\phi : \mathcal{F} \rightarrow \mathcal{G}$ , which is essentially bounded with respect to  $\nu_y$  on each  $X \times \{y\}$  will induce a natural transformation with components given by

$$\Phi_{\phi, \nu_y, \mathcal{H}} = \int^\oplus Id_{\mathcal{H}_x} \otimes \phi_{(x,y)} d\nu_y(x).$$

If we restrict our attention to measures with  $\mu_y$  is absolutely continuous with respect to  $\nu_y$  for every  $y \in Y$ , we can also use any essentially bounded field of operators to induce a natural transformation, but only after we normalize by multiplying by  $\sqrt{\frac{d\mu_y}{d\nu_y}}$ , the square root of the fiberwise Radon-Nikodym derivative of  $\mu_y$  with respect to  $\nu_y$ .

Even for pairs of totally  $\sigma$ -finite measures, for which the measure defining the source is not absolutely continuous with respect to the measure defining the target, we can apply the Lebesgue decomposition theorem to construct a natural transformation induced by any  $\nu_y$  essentially bounded field of operators: Decompose  $\mu_y$  as  $\tilde{\mu}_y + \hat{\mu}_y$ , where  $\tilde{\mu}_y$  is dominated by  $\mu_y$  and is absolutely continuous with respect to  $\nu_y$  and  $\hat{\mu}_y$  and  $\nu_y$  are singular. Now observe that for any section  $\zeta_{(x,y)}$  of  $\mathcal{H} \otimes \mathcal{K}$ , the condition that  $\{\int^\oplus \|\zeta_{x,y}\|^2 d\mu_y(x)\}^{\frac{1}{2}}$  be finite implies the same condition with  $\mu_y$  replaced with  $\tilde{\mu}_y$ . We can thus apply the construction of the previous paragraph to use any  $\nu_y$ -essentially bounded field of operators to induce a natural transformation. (Note: there is in general an non-trivial kernel (in the algebraic sense) in passing from the direct integral with respect to  $\mu_y$  to that with respect to  $\tilde{\mu}_y$ .)

**Definition 48** A measurable natural transformation between two measurable functors

$$\Phi_{\mathcal{F}, \mu_y}, \Phi_{\mathcal{G}, \nu_y} : \mathbf{Meas}(X, S) \rightarrow \mathbf{Meas}(Y, T)$$

is a natural transformation with component at  $\mathcal{H}$  given by

$$\zeta_y \mapsto \int^{\oplus} \sqrt{\frac{d\tilde{\mu}_y}{d\nu_y}} Id_{\mathcal{H}_x} \otimes B_{(x,y)}(\zeta_{x,y}) d\nu_y(x)$$

where  $\zeta_y = \int^{\oplus} \zeta_{x,y} d\mu_y(x)$ , for some field of operators  $B : \mathcal{F} \rightarrow \mathcal{G}$  which is  $\nu_y$ -essentially bounded for all  $y$ .

It is easy to see that identity natural transformations are measurable, and that both compositions of measurable natural transformations are again measurable (this latter needing the chain rule for Radon-Nikodym derivatives).

We denote the bicategory whose objects are the categories  $\mathbf{Meas}(X, S)$ , 1-arrows are all measurable functors, and 2-arrows are all measurable natural transformations by  $\mathbf{Meas}$ .

## 6 Tensor Products

We have already considered a number of constructions involving tensor products, all of which have been quite well-behaved. The following result is thus not surprising:

**Theorem 49** For any Borel space  $(X, S)$  the category  $\mathbf{Meas}(X, S)$  is a monoidal category when equipped with

$$[\mathcal{H} \otimes \mathcal{K}]_x = \mathcal{H}_x \otimes \mathcal{K}_x$$

with fundamental sequence given by all algebraic linear combinations of elements of the form  $\eta_i \otimes \kappa_j$ , where  $\{\eta_i\}$  and  $\{\kappa_j\}$  are fundamental sequences for  $\mathcal{H}$  and  $\mathcal{K}$  respectively, the measurable sections are the closure of this fundamental sequence under condition 2, and the total measurable line bundle as  $I$ . Structure maps are given fiberwise by the corresponding structure maps for Hilbert-space tensor product.

**proof:** The condition from [9] IV §8 Lemma 8.10 follows from the same argument as in Proposition 42. Coherence follows from the coherence for the structure maps in each fiber. •

What is perhaps a little more surprising is that these tensor products and the cartesian product of Borel spaces induce a monoidal bicategory structure on the 2-category  $\mathbf{Meas}$ :

**Theorem 50** ***Meas** is a monoidal bicategory when equipped with the monoidal bifunctor  $\odot$  given on objects by  $\mathbf{Meas}(X) \odot \mathbf{Meas}(Y) = \mathbf{Meas}(X \times Y)$ , and induced on 1- and 2-arrows by the functors  $\odot = p_X^*(p_1) \otimes p_Y^*(p_2) : \mathbf{Meas}(X) \times \mathbf{Meas} \rightarrow \mathbf{Meas}(X \times Y)$ , where  $p_X$  and  $p_Y$  are the projection maps from  $X \times Y$  onto its factors and  $p_1$  and  $p_2$  are the projection functors from  $\mathbf{Meas}(X) \times \mathbf{Meas}(Y)$  onto its factors, and the object  $\mathbf{1}$  as identity.*

**sketch of proof:** The structural 1-arrows are induced by the corresponding structural arrows for cartesian product of Borel spaces and tensor product of fields of Hilbert spaces. The structural 2-arrows in turn are all identities either by the coherence for the two products inducing the structural 1-arrows, or by virtue of the functoriality in each variable of the operation inducing  $\odot$ . •

## 7 Direct Integrals in $\mathbf{Meas}(X)$

The construction of measurable functors given above may be used to construct direct integrals of  $Y$ -indexed families of objects (for a Borel space  $Y$ ) in any  $\mathbf{Meas}(X)$ .

**Definition 51** *A measurable field of  $\mathbf{Meas}(X)$ -objects on a Borel space  $Y$  is a measurable field of Hilbert spaces on  $X \times Y$ . Similarly a (bounded) field of  $\mathbf{Meas}(X)$ -arrows on  $Y$  is a (bounded) field of operators on  $X \times Y$ .*

We can then define a direct integral of such a measurable field by

**Definition 52** *Given a measurable field  $\mathcal{K}$  of  $\mathbf{Meas}(X)$ -objects on  $Y$ , and a measure  $\nu$  on  $Y$ , the direct integral*

$$\int^{\oplus} \mathcal{K}_{<x,y>} d\nu(y)$$

*is the image of the total measurable line bundle  $\mathbb{C}$  on  $Y$  under the measurable functor  $\Phi_{\mathcal{K},\nu}$ , where  $\nu$  is interpreted as the  $p_1$  fibered measure for which*

$$\nu_x(A) = \nu(p_2(A \cap p_1^{-1}(x)).$$

Observe that here we have switched the role of first and second projection, however the fact that this defines a measurable field follows from the work done in the section on measurable functors.

This definition, together with the fibered measure of Example 37 allows us to give a version of Fubini's Theorem for direct integrals.

**Theorem 53** *If  $\mu$  (resp.  $\nu$ ) is a measure on the Borel space  $X$  (resp.  $Y$ ), then for any measurable field of Hilbert spaces  $\mathcal{H}$  on  $X \times Y$*

$$\int_{X \times Y}^{\oplus} \mathcal{H}_{(x,y)} d\mu(x) \times d\nu(y) \cong \int_X^{\oplus} \left\{ \int_Y^{\oplus} \mathcal{H}_{(x,y)} d\nu(y) \right\} d\mu(x).$$

**proof:** The result follows from the classical Fubini's Theorem applied to the integrals in the defining conditions for measurability and  $L^2$ -ness of sections.

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## 8 Measurable Categories and Measurable Bicategories

As observed in the introduction, our purpose in examining in detail the structure of categories of measurable fields of Hilbert spaces and of organizing them into a (monoidal) bicategory was to provide a setting for a representation theory for categorical groups.

That theory is developed in [4] and [3]. It will turn out to be the first example of what should be a general theory of measurable bicategories. Our purpose in this final section is to suggest the outline of general theories of measurable categories over a measure space  $X$  and of measurable bicategories which are analogues of Tannakian categories, but with  $\mathbf{VECT}$  replaced by  $\mathbf{Meas}(X)$  and  $\mathbf{Meas}$  respectively.

**Definition 54** A measurable category over  $X$  is a monoidal category  $\mathcal{C}$  equipped with monoidal functors  $U : \mathcal{C} \rightarrow \mathbf{Meas}(X)$  and  $T : \mathbf{Meas}(X) \rightarrow \mathcal{C}$  such that there is a natural isomorphism  $U(T) \cong \text{Id}_{\mathbf{Meas}(X)}$ . Objects of  $\mathcal{C}$  isomorphic to object of the form  $T(\mathcal{H})$  are called trivial objects.

**Example 55**  $\mathbf{Meas}(X)$  with both structural functors being the identity functor.

**Example 56** Fix a Lie group  $G$ , let  $\mathbf{Rep}(G)/X$  be the category whose objects are measurable fields of Hilbert spaces on  $X$  such that each fiber is equipped with a unitary representation of  $G$ , and moreover for each  $g \in G$   $g$  acts by a bounded field of bounded operators. Then the forgetful functor to underlying measurable fields and the inclusion on measurable fields with a trivial  $G$ -action on each fiber make  $\mathbf{Rep}(G)/X$  into a measurable category over  $X$ .

Of more interest is the general setting in which the motivating construction fits:

**Definition 57** A measurable bicategory  $\mathcal{C}$  is a monoidal bicategory  $\mathcal{C}$  equipped with monoidal bifunctors  $U : \mathcal{C} \rightarrow \mathbf{Meas}$  and  $T : \mathbf{Meas} \rightarrow \mathcal{C}$  such that  $U(T)$  is naturally isomorphic to  $\text{Id}_{\mathbf{Meas}}$

Of course  $\mathbf{Meas}$  itself gives a tautological example.

The subject of [4] gives others:

**Example 58** Let  $\mathcal{G}$  be a categorical group. Regarding  $\mathcal{G}$  as a bicategory with one object, the functor bicategory  $\mathbf{Meas}^{\mathcal{G}}$  (with bifunctors as objects, pseudonatural transformations as 1-arrows, and modifications as 2-arrows) is a measurable bicategory (with monoidal structure induced by  $\odot$  in an obvious way).

The subcategories considered in [4] provide more examples.

Somewhat curiously, letting the underlying Borel space vary, measurable categories themselves can be organized into a measurable bicategory:

**Example 59**  $\mathbf{MeasCat}$  is a measurable bicategory whose objects are all measurable categories; whose 1-arrows from  $\mathcal{C}$  to  $\mathcal{D}$  are pairs of monoidal functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $\Phi : \mathbf{Meas}(X) \rightarrow \mathbf{Meas}(Y)$  with  $\Phi$  measurable and such that both the obvious square formed by  $F$ ,  $\Phi$  and the underlying functors and the obvious square formed by  $F$ ,  $\Phi$  and the inclusion of trivial object functors commute; and whose 2-arrows from  $(F, \Phi)$  to  $(G, \Gamma)$  are pairs of 2-arrows  $(h : F \Rightarrow G, \eta : \Phi \Rightarrow \Gamma)$  such that the two obvious pillows commute.

The underlying functor simply assigns  $\mathbf{Meas}(X)$  to the measurable category  $(\mathcal{C}, \mathbf{Meas}(X), U, T)$ , while the inclusion of trivial objects assigns to  $\mathbf{Meas}(X)$ , the tautological  $(\mathbf{Meas}(X), \mathbf{Meas}(X), Id, Id)$ .

The monoidal structure on  $\mathbf{MeasCat}$  is given by letting  $\mathcal{C} \odot \mathcal{D}$ , for  $\mathcal{C}$  (resp.  $\mathcal{D}$ ) a measurable category over  $X$  (resp.  $Y$ ), be the measured category over  $X \times Y$  with objects given by a pair of a  $Y$ -indexed family of  $\mathcal{C}$  objects whose underlying  $Y$ -indexed family of fields of Hilbert spaces on  $X$  forms a measurable field of Hilbert spaces on  $X \times Y$  and an  $X$ -indexed family of  $\mathcal{D}$  objects whose underlying  $X$ -indexed family of fields of Hilbert spaces on  $Y$  forms a measurable field of Hilbert spaces on  $X \times Y$ . The underlying field of Hilbert spaces for an object  $(C_y, D_x)$  is the tensor product of the two underlying fields of the entries. The arrows are generated by the family similarly defined, and by formally adjointed isomorphisms between  $(C_y \otimes T_{\mathcal{C}}^Y(\mathcal{H}_{<x, y>}, D_x)$  and  $(C_y, T_{\mathcal{D}}^X(\mathcal{H}_{<x, y> \otimes D_x})$ , where  $T_{\mathcal{C}}^Y$  (resp.  $T_{\mathcal{D}}^X$ ) is the trivial functor for  $\mathcal{C}$  (resp.  $\mathcal{D}$ ) applied in each of fiber over  $Y$  (resp.  $X$ ).

## A Elementary results on Hilb

One problem in approaching this work is a paucity of references on the categorical structure of the category of Hilbert spaces and bounded operators. The results in this appendix are elementary and, in non-categorical guise, classical. We include them for completeness of exposition, but in an appendix so as not to interrupt the flow of the new results.

**Definition 60** The category of separable Hilbert spaces  $\mathbf{Hilb}$  has as objects all separable Hilbert spaces and as arrows all bounded operators. Source, target, identities and composition are obvious.

Many facts about  $\mathbf{Hilb}$  are set forth in [5].

Of these, the most important for us is the fact that  $\mathbf{Hilb}$  is a  $C^*$ -category. It is thus *a fortiori* equivalent to its opposite category by an equivalence which is the identity on objects, adjoint-operator being the functor in the equivalence in either direction.

Ghez, Lima and Roberts [5], do not, however address some of the elementary category-theoretic properties we will use. We summarize these in

**Proposition 61**  $\mathbf{Hilb}$  is a  $\mathbb{C}$ -linear additive category with all finite limits and colimits.

**proof:**  $\mathbb{C}$ -linearity is obvious. To see that **Hilb** is additive, observe that the 0-dimensional vectorspace is a Hilbert space and is a zero (initial and terminal) object for **Hilb**, likewise the vector space direct sum of two Hilbert spaces, equipped with the scalar product  $\langle(\xi, \phi)|(\zeta, \psi)\rangle = \langle\xi, \zeta\rangle + \langle\phi, \psi\rangle$  is a Hilbert space, and all of the structural maps for it as a vector-space biproduct are bounded operators. Thus it is a biproduct in **Hilb**.

By self-duality, it suffices to show that **Hilb** has all finite limits, and by standard results, it suffices to show that it has kernels and binary products. Kernels are easy: the vector-space kernel will again be a Hilbert space with the scalar product inherited from the source of the operator. (Note: since bounded operators are continuous, the kernel will be closed, and thus complete.) Finite products follow from biproducts. •

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